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Horizon wave function for single localized particles: GUP and quantum black-hole decay

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Abstract A localized particle in Quantum Mechanics is described by a wave packet in position space, regardless of its energy. However, from the point of view of General Relativity, if the particle's energy density exceeds a certain threshold, it should be a black hole. To combine these two pictures, we introduce a horizon wave function determined by the particle wave function in position space, which eventually yields the probability that the particle is a black hole. The existence of a minimum mass for black holes naturally follows, albeit not in the form of a sharp value around the Planck scale, but rather like a vanishing probability that a particle much lighter than the Planck mass may be a black hole. We also show that our construction entails an effective generalized uncertainty principle (GUP), simply obtained by adding the uncertainties coming from the two wave functions associated with a particle. Finally, the decay of microscopic (quantum) black holes is also described in agreement with what the GUP predicts.

1 Introduction and motivation

Understanding all the physical aspects in the gravitational collapse of a compact object, and how black holes form, remains one of the most intriguing challenges of contemporary theoretical physics. After the seminal papers of Oppenheimer et al. [1,2], the literature on the subject has grown immensely, but many issues are still open in General Relativity (see, e.g. Refs. [3,4], and references therein), not to mention the conceptual and technical difficulties one faces when the quantum nature of the collapsing matter is taken into account. Assuming that quantum gravitational fluctuations are small, one can describe matter by means of Quantum Field Theory on a curved background space-time, an approach which has produced remarkable results but is unlikely to be directly applicable to a self-gravitating system representing a collapsing object.

A general property of the Einstein theory is that the gravitational interaction is always attractive and we are thus not allowed to neglect its effect on the causal structure of space-time if we pack enough energy in a sufficiently small volume. This can occur, for example, if two particles (for simplicity of negligible spatial extension and total angular momentum) collide with an impact parameter b shorter than the Schwarzschild radius corresponding to the total center-mass energy E of the system, that is[1],

$$b \lesssim 2\,\ell_\mathrm{p}\,\frac{E}{m_\mathrm{p}} \equiv R_\mathrm{H}. \tag{1}$$

This *hoop conjecture* [5] has been checked and verified theoretically in a variety of situations, but it was initially formulated for black holes of (at least) astrophysical size [6–8], for which the very concept of a classical background metric and related horizon structure should be reasonably safe (for a review of some problems, see the bibliography in Ref. [9]). Whether the concepts involved in the above conclusion can also be trusted for masses approaching the Planck size, however, is definitely more challenging. In fact, for masses in that range, quantum effects may hardly be neglected (for a recent discussion, see, e.g. Ref. [10]) and it is reasonable that the picture arising from General Relativistic black holes must be replaced to include the possible existence of new objects, generically referred to as "quantum black holes" (see, e.g. Refs. [11–13]).

The main complication in studying the Planck regime is that we do not have any experimental insight thereof, which

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[1] We shall use units with $c = k_B = 1$, and always display the Newton constant $G = \ell_\mathrm{p}/m_\mathrm{p}$, where ℓ_p and m_p are the Planck length and mass, respectively, so that $\hbar = \ell_\mathrm{p}\,m_\mathrm{p}$.

makes it very difficult to tell whether any theory we could come up with is physically relevant. We might instead start from our established concepts and knowledge of nature, and push them beyond the present experimental limits. If we set out to do so, we immediately meet with a conceptual challenge: how can we describe a system containing both Quantum Mechanical objects (such as the elementary particles of the Standard Model) and classically defined horizons? The aim of this paper is precisely to introduce the definition of a wave function for the horizon that can be associated with any localized Quantum Mechanical particle [14]. This tool will allow us to put on quantitative ground the condition that distinguishes a black hole from a regular particle. We shall also see that our construction naturally leads to an effective Generalized Uncertainty Principle (GUP) [15–19] for the particle position, and a decay rate for microscopic black holes.

The paper is organized as follows: in the next section, we introduce the main ideas that define the horizon wave function associated with any localized Quantum Mechanical particle; in Sect. 3, we then apply the general construction to the particularly simple case of a particle described by a Gaussian wave function at rest in flat space-time, for which we explicitly obtain the probability that the particle is a black hole, we recover the GUP and a minimum measurable length, and estimate the decay rate of a black hole with mass around the Planck scale; finally, in Sect. 4, we comment on our findings and outline future applications.

2 Horizon Quantum Mechanics

Given a matter source, say a spherically symmetric "particle", General Relativity and Quantum Mechanics naturally associate with it two length scales: the Schwarzschild radius and the Compton–de Broglie wavelength, respectively. We shall, therefore, start by briefly reviewing these concepts and then propose how to extend the former into the realm of Quantum Mechanics, where the latter is born.

2.1 Spherical trapping horizons

The appearance of a classical horizon is relatively easy to understand in a spherically symmetric space-time. Let us first recall that we can write a general spherically symmetric metric $g_{\mu\nu}$ as

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + r^2(x^i)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{2}$$

where r is the areal coordinate and $x^i = (x^1, x^2)$ are coordinates on surfaces where the angles θ and ϕ are constant. The location of a trapping horizon, a surface where the escape velocity equals the speed of light[2], is then determined by the equation [20]

$$0 = g^{ij}\,\nabla_i r\,\nabla_j r = 1 - \frac{2\,M}{r}, \tag{3}$$

where $\nabla_i r$ is the covector perpendicular to the surfaces of constant area $\mathcal{A} = 4\pi r^2$. The function $M = \ell_\mathrm{p}\, m/m_\mathrm{p}$ is the active gravitational (or Misner–Sharp) mass, representing the total energy enclosed within a sphere of radius r. For example, if we set $x^1 = t$ and $x^2 = r$, the function m is explicitly given by the integral of the classical matter density $\rho = \rho(x^i)$ weighted by the flat metric volume measure,

$$m(t, r) = \frac{4\pi}{3}\int_0^r \rho(t, \bar{r})\,\bar{r}^2\,\mathrm{d}\bar{r}, \tag{4}$$

as if the space inside the sphere were flat. Of course, it is, in general, very difficult to follow the dynamics of a given matter distribution and verify the existence of surfaces satisfying Eq. (3), but we can say an horizon exists if there are values of r such that

$$R_M = 2\,M(t, r) \geq r, \tag{5}$$

which generalizes the hoop conjecture (1) to continuous energy densities (in fact, the horizon radius saturates the above inequality, i.e. $R_\mathrm{H} = r$).

Note that the above equation does not lead to any mass threshold for the existence of a black hole, since M is not limited from below in the classical theory, and the area of the trapping surface can be vanishingly small. However, if we consider a spin-less point-like source of mass m, Quantum Mechanics introduces an uncertainty in its spatial localization, typically of the order of the Compton length:

$$\lambda_m \simeq \ell_\mathrm{p}\,\frac{m_\mathrm{p}}{m} = \frac{\ell_\mathrm{p}^2}{M}. \tag{6}$$

Assuming that quantum physics is a more refined description of classical physics, the clash of the two lengths R_H and λ_m implies that the former only makes sense provided that it is larger than the latter,

$$R_\mathrm{H} \gtrsim \lambda_m \quad \Rightarrow \quad m \gtrsim m_\mathrm{p}, \tag{7}$$

or $M \gtrsim \ell_\mathrm{p}$. Note that this argument employs the flat space Compton length (6), and it is likely that the particle's self-gravity will affect it. However, it is still reasonable to assume that the condition (7) holds as an order of magnitude estimate.

Overall, the common argument that quantum gravity effects should become relevant only at scales of order m_p or higher now appears questionable, since the condition (7) implies that such a system can be fairly well described in

[2] More technically, a trapping surface is the location where the divergence of outgoing null congruences vanishes.

classical terms. This is indeed at the core of the idea of "classicalization" given in Ref. [21,22] and, before that, of gravitationally inspired GUPs [15–19,23]. In particular, following the canonical steps that lead to the construction of Quantum Mechanics, the latter are usually assumed to hold as fundamental principles for the reformulation of Quantum Mechanics in the presence of gravity. Note that gravity would reduce to a "kinematical effect" encoded by the modified commutators for the canonical variables. In the following, we shall instead start from the introduction of an auxiliary wave function that describes the horizon associated with a given localized particle, and we show that a modified uncertainty relation follows consequently.

2.2 Horizon wave function

Let us first formulate the construction in a somewhat general fashion. For simplicity, we shall only consider quantum mechanical states representing *spherically symmetric* objects, which are both *localized in space* and *at rest* in the chosen reference frame. The particle is consequently described by a wave function $\psi_{\rm S} \in L^2(\mathbb{R}^3)$, which we assume can be decomposed into energy eigenstates,

$$|\psi_{\rm S}\rangle = \sum_E C(E)\,|\psi_E\rangle, \tag{8}$$

where the sum represents the spectral decomposition in Hamiltonian eigenmodes,

$$\hat{H}\,|\psi_E\rangle = E\,|\psi_E\rangle, \tag{9}$$

and the actual Hamiltonian H needs not be specified yet[3]. The expression of the Schwarzschild radius in Eq. (1) can be inverted to obtain

$$E = m_{\rm p}\,\frac{R_{\rm H}}{2\,\ell_{\rm p}}, \tag{10}$$

and we then define the (unnormalized) "horizon wave function" as $\tilde{\psi}_{\rm H}(R_{\rm H}) = C(m_{\rm p}\,R_{\rm H}/2\,\ell_{\rm p})$, whose normalization is fixed by assuming the scalar product

$$\langle\,\psi_{\rm H}|\phi_{\rm H}\,\rangle = 4\,\pi \int_0^\infty \psi_{\rm H}^*(R_{\rm H})\,\phi_{\rm H}(R_{\rm H})\,R_{\rm H}^2\,{\rm d}R_{\rm H}. \tag{11}$$

We could now simply say that the normalized wave function $\psi_{\rm H}$ yields the probability that an observer would detect a horizon of areal radius $r = R_{\rm H}$ associated with the particle in the quantum state $\psi_{\rm S}$. Such a horizon would be necessarily "fuzzy", like the position of the particle itself, but giving such a claim experimental meaning does not appear to be very simple.

[3] This is where, for instance, the self-gravity of the particle may enter.

A more precise use of the notion of the horizon wave function can, however, already be outlined. For example, having defined the wave function $\psi_{\rm H}$ associated with a given $\psi_{\rm S}$, the probability density that the particle lies inside its own horizon of radius $r = R_{\rm H}$ will be given by

$$P_<(r < R_{\rm H}) = P_{\rm S}(r < R_{\rm H})\,P_{\rm H}(R_{\rm H}), \tag{12}$$

where

$$P_{\rm S}(r < R_{\rm H}) = 4\,\pi \int_0^{R_{\rm H}} |\psi_{\rm S}(r)|^2\,r^2\,{\rm d}r \tag{13}$$

is the probability that the particle is inside a sphere of radius $r = R_{\rm H}$, and

$$P_{\rm H}(R_{\rm H}) = 4\,\pi\,R_{\rm H}^2\,|\psi_{\rm H}(R_{\rm H})|^2 \tag{14}$$

is the probability that the sphere of radius $r = R_{\rm H}$ is a horizon. Finally, the probability that the particle described by the wave function $\psi_{\rm S}$ is a black hole will be obtained by integrating (12) over all possible values of the horizon radius, namely

$$P_{\rm BH} = \int_0^\infty P_<(r < R_{\rm H})\,{\rm d}R_{\rm H}. \tag{15}$$

It is this final probability we now proceed to clarify with an example, along with a derivation of a GUP and some predictions for the decay of a quantum black hole.

3 Gaussian packet at rest in flat space

Assuming that the space-time is flat, our construction can be exemplified by describing the massive particle at rest in the origin of the reference frame with the spherically symmetric Gaussian wave function

$$\psi_{\rm S}(r) = \frac{{\rm e}^{-\frac{r^2}{2\,\ell^2}}}{\ell^{3/2}\,\pi^{3/4}}, \tag{16}$$

where we shall usually assume that the width ℓ is given by the Compton length (6) of the particle

$$\ell = \lambda_m \simeq \ell_{\rm p}\,\frac{m_{\rm p}}{m}. \tag{17}$$

The above packet corresponds to the momentum space wave function

$$\psi_{\rm S}(p) = \frac{{\rm e}^{-\frac{p^2}{2\,\Delta^2}}}{\Delta^{3/2}\,\pi^{3/4}}, \tag{18}$$

where $p^2 = \vec{p}\cdot\vec{p}$ is the square modulus of the spatial momentum, and the width

$$\Delta = m_{\rm p}\,\frac{\ell_{\rm p}}{\ell} \simeq m. \tag{19}$$

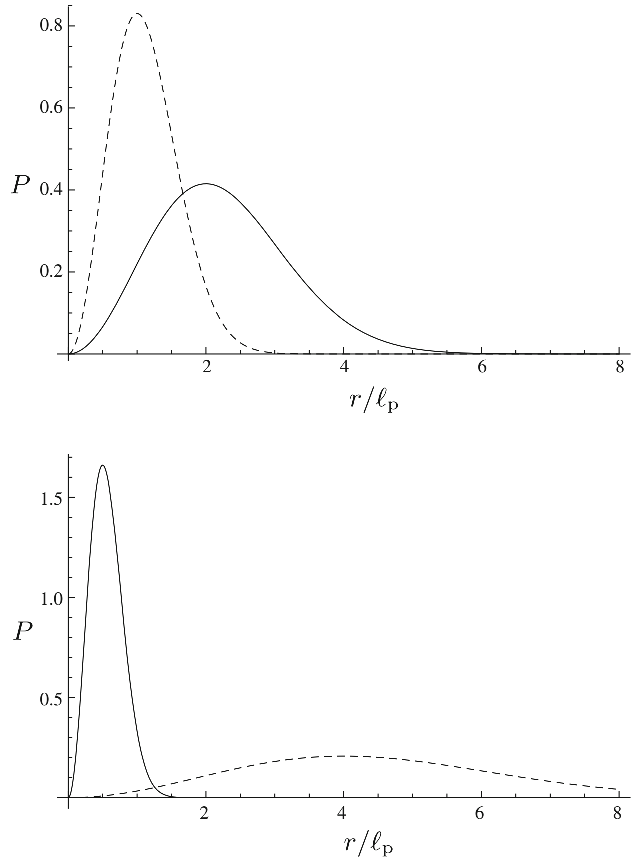

Fig. 1 Probabilities $P_{\rm H}$ in Eq. (14) (*dashed line*) and $P_{\rm S}$ in Eq. (23) (*solid line*) for $m = m_{\rm p}/2$ (*upper panel*) and $m = 2\,m_{\rm p}$ (*lower panel*), assuming $m \sim \ell^{-1}$

For the energy of the particle, we can simply assume the relativistic mass-shell relation in flat space

$$E^2 = p^2 + m^2, \tag{20}$$

and, upon inverting the expression of the Schwarzschild radius (1), we obtain the unnormalized wave function

$$\tilde{\psi}_{\rm H}(R_{\rm H}) = \frac{\ell^{3/2}\, e^{\frac{\ell^2 m^2}{2\,\ell_{\rm p}^2 m_{\rm p}^2}}\, e^{-\frac{\ell_{\rm p}^2 R_{\rm H}^2}{8\,\ell_{\rm p}^4}}}{\pi^{3/4}\,\ell_{\rm p}^{3/2}\, m_{\rm p}^{3/2}}. \tag{21}$$

Finally, the inner product (11) yields the normalized horizon wave function

$$\psi_{\rm H}(R_{\rm H}) = \frac{\ell^{3/2}\, e^{-\frac{\ell^2 R_{\rm H}^2}{8\,\ell_{\rm p}^4}}}{2^{3/2}\,\pi^{3/4}\,\ell_{\rm p}^3}. \tag{22}$$

Note that, since $\langle\, \hat{r}^2 \,\rangle \simeq \ell^2$ and $\langle\, \hat{R}_{\rm H}^2 \,\rangle \simeq \ell_{\rm p}^4/\ell^2$, we expect that the particle will be inside its own horizon if $\langle\, \hat{r}^2 \,\rangle \ll \langle\, \hat{R}_{\rm H}^2 \,\rangle$, which precisely yields the condition (7) if $\ell \sim m^{-1}$. This is clear, for example, in Fig. 1, where the probability $P_{\rm H} = P_{\rm H}(r)$ is plotted along with the probability

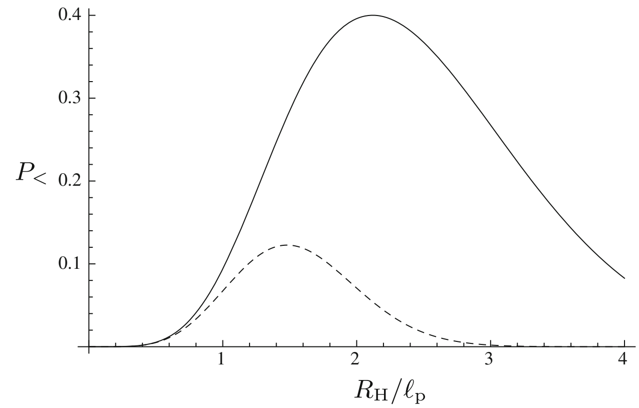

Fig. 2 Probability density $P_<$ in Eq. (24) that particle is inside its horizon of radius $R = R_{\rm H}$, for $\ell = \ell_{\rm p}$ (*solid line*) and for $\ell = 2\,\ell_{\rm p}$ (*dashed line*)

$$P_{\rm S}(r) = 4\,\pi\, r^2\, |\psi_{\rm S}(r)|^2, \tag{23}$$

for $m \lesssim m_{\rm p}$ and $m \gtrsim m_{\rm p}$. In the former case, the horizon is more likely found with a smaller radius than the particle's, with the opposite occurring in the latter case. In fact, the probability density (12) can now explicitly be computed:

$$P_< = \frac{\ell^3\, R_{\rm H}^2\, e^{-\frac{\ell^2 R_{\rm H}^2}{4\,\ell_{\rm p}^4}}}{2\,\sqrt{\pi}\,\ell_{\rm p}^6} \left[{\rm Erf}\left(\frac{R_{\rm H}}{\ell}\right) - \frac{2\,R_{\rm H}\, e^{-\frac{R_{\rm H}^2}{\ell^2}}}{\sqrt{\pi}\,\ell} \right], \tag{24}$$

from which the probability (15) for the particle to be a black hole is obtained as

$$P_{\rm BH}(\ell) = \frac{2}{\pi} \left[\arctan\left(2\,\frac{\ell_{\rm p}^2}{\ell^2}\right) + 2\,\frac{\ell^2\,(4 - \ell^4/\ell_{\rm p}^4)}{\ell_{\rm p}^2\,(4 + \ell^4/\ell_{\rm p}^4)^2} \right], \tag{25}$$

or, writing $P_{\rm BH}$ as a function of m,

$$P_{\rm BH} = \frac{2}{\pi} \left[\arctan\left(2\,\frac{m^2}{m_{\rm p}^2}\right) + 2\,\frac{m_{\rm p}^2\,(4 - m_{\rm p}^4/m^4)}{m^2\,(4 + m_{\rm p}^4/m^4)^2} \right]. \tag{26}$$

In Fig. 2, we show the probability density (24), for two different values of the Gaussian width ℓ. Since $\ell \sim m^{-1}$, it is already clear that such a probability decreases for decreasing m (below the Planck mass). In fact, in Fig. 3, we show the probability (25) that the particle is a black hole as a function of the Gaussian width ℓ (upper panel) and particle mass $m \sim \ell^{-1}$ (lower panel). From the plot of $P_{\rm BH}$, it is pretty obvious that the particle is most likely a black hole, $P_{\rm BH} \simeq 1$, if $\ell \lesssim \ell_{\rm p}$. Assuming as usual $\ell \sim m^{-1}$, we have thus derived the same condition (7) from a totally Quantum Mechanical picture.

An important remark is that we have here assumed flat space throughout the computation, which means that the self-gravity of the particle has been neglected. It is very likely

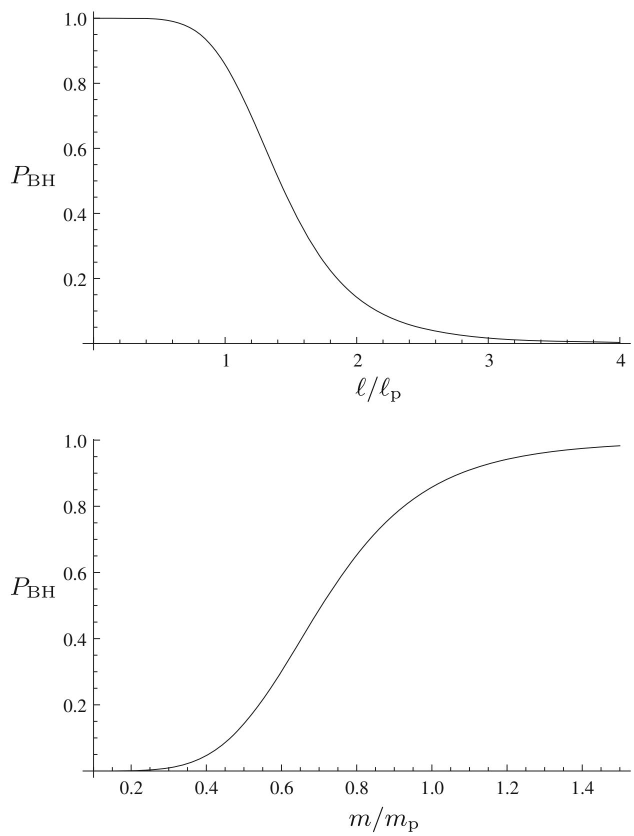

Fig. 3 Probability $P_{\rm BH}$ in Eq. (25) that particle of width $\ell \sim m^{-1}$ is a black hole

that such an approximation fails for large black holes with $m \gg m_{\rm p}$, although the general idea outlined in Sect. 2.2 should still be valid. Of course, one could then improve the description of particles with $m \gg m_{\rm p}$ by employing a curved-space mass-shell relation and suitable normal modes, rather than simple plane waves.

3.1 Effective GUP

For the Gaussian packet described above, it is easy to find that the usual Quantum Mechanical uncertainty in radial position is given by

$$\begin{aligned}\langle \Delta r^2 \rangle &= 4\,\pi \int_0^\infty |\psi_{\rm S}(r)|^2\, r^4\, {\rm d}r \\ &\quad - \left(4\,\pi \int_0^\infty |\psi_{\rm S}(r)|^2\, r^3\, {\rm d}r\right)^2 \\ &= \left(\frac{3\,\pi - 8}{2\,\pi}\right) \ell^2. \end{aligned} \tag{27}$$

Analogously, the uncertainty in the horizon radius will be given by

$$\begin{aligned}\langle \Delta R_{\rm H}^2 \rangle &= 4\,\pi \int_0^\infty |\psi_{\rm H}(R_{\rm H})|^2\, R_{\rm H}^4\, {\rm d}R_{\rm H} \\ &\quad - \left(4\,\pi \int_0^\infty |\psi_{\rm H}(R_{\rm H})|^2\, R_{\rm H}^3\, {\rm d}R_{\rm H}\right)^2 \\ &= 4\left(\frac{3\,\pi - 8}{2\,\pi}\right) \frac{\ell_{\rm p}^4}{\ell^2}. \end{aligned} \tag{28}$$

Since

$$\begin{aligned}\langle \Delta p^2 \rangle &= 4\,\pi \int_0^\infty |\psi_{\rm S}(p)|^2\, p^4\, {\rm d}p \\ &\quad - \left(4\,\pi \int_0^\infty |\psi_{\rm S}(p)|^2\, p^3\, {\rm d}p\right)^2 \\ &= \left(\frac{3\,\pi - 8}{2\,\pi}\right) m_{\rm p}^2\, \frac{\ell_{\rm p}^2}{\ell^2} \equiv \Delta p^2, \end{aligned} \tag{29}$$

we can also write

$$\ell^2 = \left(\frac{3\,\pi - 8}{2\,\pi}\right) \ell_{\rm p}^2\, \frac{m_{\rm p}^2}{\Delta p^2}. \tag{30}$$

Finally, by combining the uncertainty (27) with (28) linearly, we find

$$\begin{aligned}\Delta r &\equiv \sqrt{\langle \Delta r^2 \rangle} + \gamma\, \sqrt{\langle \Delta R_{\rm H}^2 \rangle} \\ &= \left(\frac{3\,\pi - 8}{2\,\pi}\right) \ell_{\rm p}\, \frac{m_{\rm p}}{\Delta p} + 2\,\gamma\, \ell_{\rm p}\, \frac{\Delta p}{m_{\rm p}}, \end{aligned} \tag{31}$$

where γ is a coefficient of order 1, and the result is plotted in Fig. 4 (for $\gamma = 1$). This is precisely the kind of GUP considered in Refs. [15–19], leading to a minimum measurable length

$$\Delta r \geq 2\,\sqrt{\gamma\, \frac{3\,\pi - 8}{\pi}}\, \ell_{\rm p} \simeq 1.3\,\sqrt{\gamma}\, \ell_{\rm p}, \tag{32}$$

obtained for

$$\Delta p = \sqrt{\frac{3\,\pi - 8}{\pi\,\gamma}}\, \frac{m_{\rm p}}{2}. \tag{33}$$

Of course, one might consider different ways of combining the two uncertainties (27) and (28), or even avoid this step and just make direct use of the horizon wave function. In this respect, the present approach appears to be more flexible, provided that one is able to extend it to different physical systems, as we shall further discuss in the last section.

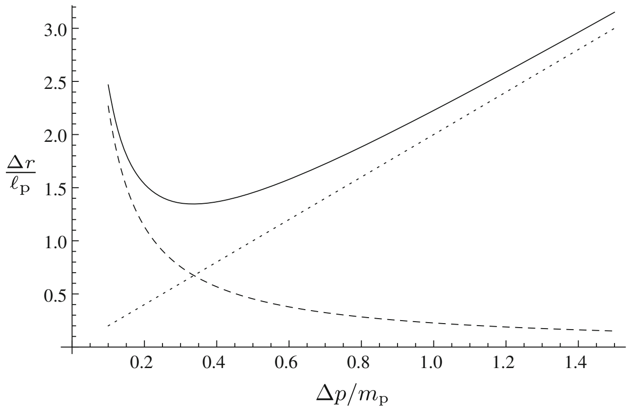

Fig. 4 Uncertainty relation (31) (*solid line*) as a combination of the Quantum Mechanical uncertainty (*dashed line*) and the uncertainty in horizon radius (*dotted line*)

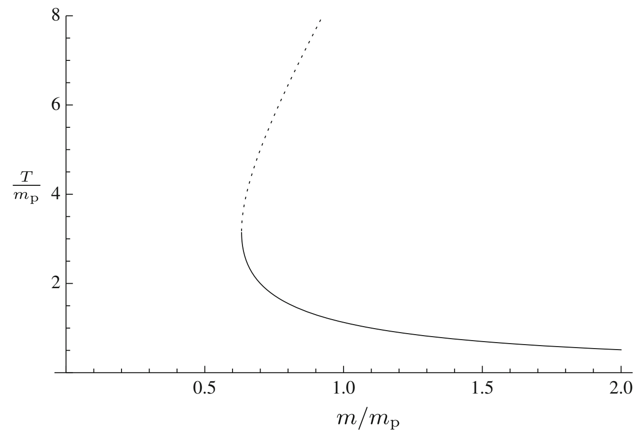

Fig. 5 Temperature vs. mass according to Eq. (35) with $\beta = 1/10$: *solid line* reproduces the Hawking behavior for large $m \gg m_{\rm p}$; *dotted line* is the unphysical branch, and their meeting point represents the black hole with minimum mass

3.2 Quantum black-hole evaporation

The well-known result due to Hawking [24,25],

$$T_{\rm H} = \frac{m_{\rm p}^2}{8\,\pi\,m}, \tag{34}$$

extrapolated to vanishingly small mass M, implies that $T_{\rm H}$ diverges. On the other hand, one can derive modified black-hole temperatures for $m \simeq m_{\rm p}$ from the GUP [26–33]. In particular, we just recall that one obtains

$$m = \frac{m_{\rm p}^2}{8\,\pi\,T} + 2\,\pi\,\beta\,T, \tag{35}$$

where

$$\beta = \frac{\gamma}{4\,\pi\,(3\,\pi - 8)} > 0, \tag{36}$$

to ensure the existence of a minimum mass for the black hole (see Fig. 5). This is a consistency condition with the result that $P_{\rm BH} \simeq 1$ only for $m \gtrsim m_{\rm p}$, or that one does not have a black hole for masses significantly smaller than $m_{\rm p}$. In fact, from (35) we get

$$m_{\rm min} = \sqrt{\beta}\,m_{\rm p}, \quad T_{\rm max} = \frac{m_{\rm p}}{4\,\pi\,\sqrt{\beta}}. \tag{37}$$

Upon solving Eq. (35), and considering the "physical" branch (which reproduces the Hawking behavior for $m \gg m_{\rm p}$), one obtains

$$\begin{aligned} T &= \frac{1}{4\,\pi\beta}\left(m - \sqrt{m^2 - \beta\,m_{\rm p}^2}\right) \\ &= \frac{1-\sqrt{1-\beta}}{4\,\pi\beta}\left(m_{\rm p} - \frac{m - m_{\rm p}}{\sqrt{1-\beta}}\right) + \mathcal{O}\left[(m - m_{\rm p})^2\right], \end{aligned} \tag{38}$$

for $0 < \beta < 1$, where we expanded around $m \simeq m_{\rm p}$. It is interesting to note that such an expression for T is still meaningful for $\beta < 0$. These possibilities hint at a lattice microstructure of the space-time, and they have been explored, e.g. in polymer quantization, and in world-crystal physics [33].

Recalling now that the emission rate can be written as

$$\frac{{\rm d}m}{{\rm d}t} = -\frac{8\,\pi^3\,m^2\,T^4}{15\,m_{\rm p}^5\,\ell_{\rm p}}, \tag{39}$$

we obtain the decay rate

$$-\frac{{\rm d}m}{{\rm d}t} \simeq \alpha\,\frac{m^2}{m_{\rm p}\,\ell_{\rm p}} + \mathcal{O}(m - m_{\rm p}), \tag{40}$$

for $T \simeq T_{\rm p} = m_{\rm p}$ (or $m \simeq m_{\rm p}$), where $4 \times 10^{-5} < \alpha < 7 \times 10^{-4}$ when $0 < \beta < 1$.

It is perhaps questionable that objects with a mass of the order of $m_{\rm p}$ can be described by the usual thermodynamical arguments, which stem from a (semi-)classical picture of black holes. However, the horizon wave function for a particle was precisely conceived to describe this quantum regime, and we can now assume that the probability the black-hole decays is given by the probability $P_{\rm T}$ that the particle can be found outside its own horizon[4]. Of course, if the mass $m \ll m_{\rm p}$, the horizon wave function tells us that the particle is most likely not a black hole to begin with, so the above interpretation must be restricted to $m \simeq m_{\rm p}$ (see again Fig. 1).

We first define

$$P_{>}(r > R_{\rm H}) = P_{\rm S}(r > R_{\rm H})\,P_{\rm H}(R_{\rm H}), \tag{41}$$

[4] The subscript T is for tunneling, which is reminiscent of the interpretation of the Hawking emission as a tunneling process through the horizon [34,35]. Note, however, that the horizon is fuzzy in our description and not a (backreacting) classical surface.

where now

$$P_S(r > R_H) = 4\pi \int_{R_H}^{\infty} |\psi_S(r)|^2 r^2 \, dr. \quad (42)$$

Upon integrating the above probability over all values of R_H, we then obtain (since $m \sim \ell^{-1}$)

$$P_T(m) = 1 - P_{BH}(m), \quad (43)$$

and, expanding (26) for $m \simeq m_p$,

$$P_T(m) \simeq a - b \, \frac{m - m_p}{m_p}, \quad (44)$$

where $a \simeq 0.14$ and $b \simeq 0.65$ are positive constants of order 1. The amount of the particle's energy that can be found outside the horizon could thus be estimated by

$$\Delta m \simeq m \, P_T \simeq a \, m + \mathcal{O}(m - m_p). \quad (45)$$

At the same time, from the time-energy uncertainty relation

$$\Delta E \, \Delta t \simeq m_p \, \ell_p, \quad (46)$$

one obtains the typical emission time

$$\Delta t \simeq \frac{\ell_p^2}{\Delta R_H} \simeq \ell, \quad (47)$$

where we used Eqs. (1) and (28). Putting the two pieces together, we then find that a near-Planck size black hole would emit according to

$$\begin{aligned} -\frac{\Delta m}{\Delta t} &\simeq a \, \frac{m}{\ell} + \mathcal{O}(m - m_p) \\ &\simeq a \, \frac{m^2}{m_p \, \ell_p} + \mathcal{O}(m - m_p), \end{aligned} \quad (48)$$

in functional agreement with the prediction from the GUP given in Eq. (40).

It is now important to remark that there is a fairly large numerical discrepancy between the numerical coefficients in Eq. (40) and those in Eq. (48). For once, this disparity can perhaps be traced back to the fact that, with Eq. (39), we are applying the canonical formalism to a Planck mass particle, which is not completely sensible, since the particle/black hole should be in quasi-equilibrium with its radiation for thermodynamical arguments to hold. The horizon wave function, instead, knows nothing of the thermodynamics, and, therefore, should have a more general validity. However, we must point out that the above description of black-hole evaporation relies on a totally static representation of the Quantum Mechanical particle, and is, therefore, to be viewed as a first attempt at modeling the decay of a quantum black hole in the present picture. A more accurate account of the microscopic structure of quantum black holes is indeed likely to change the details (see, e.g. Refs. [36–40]), but the fact that this simple treatment leads to results similar to those following from the GUP is already intriguing, and suggestive that an even more accurate Quantum Mechanical description should be possible. Finally, let us mention that in this Planckian regime, regardless of the microscopic model, it would certainly be more appropriate to use the microcanonical formalism [41,42] (based on energy conservation, a property not entailed by the GUP). Future work will be devoted to refining of the calculations in all of these directions.

4 Conclusions and outlook

We have here introduced a horizon wave function as a tool that allows us to effectively describe the emergence of a horizon in a localized Quantum Mechanical system. For the simple case of a spherically symmetric massive particle, the horizon wave function already supports the existence of a minimum black-hole mass, without assuming a priori the existence of a minimum (fundamental) length [23,43–47][5]. Moreover, it does so in a genuinely Quantum Mechanical fashion, since it produces a negligible probability that a particle with mass much smaller than m_p is a black hole, rather than giving a sharp value for the particle mass above which the transition from particle to black hole occurs. Further, the description of black holes that the horizon wave function entails was shown to be compatible with GUPs, since it yields the same kind of uncertainty relation in phase space, and a similar decay rate for Planck-size objects.

The results presented here, however, should be viewed as preliminary, as the notion of a horizon wave function requires a thorough generalization before it can be effectively employed to analyze more interesting physical problems. We already mentioned in the Introduction that it is of particular conceptual interest to study the possibility of black-hole production in high-energy collisions [53–57]. Let us here recall that, along these lines, Dvali et al. [21,22] recently conjectured that the high-energy limit of all physically relevant Quantum Field Theories involves the formation of a (semi)classical state (to wit, black-hole formation for gravity), which should automatically suppress trans-Planckian quantum fluctuations. This idea extends the concept of a GUP to include gravity, as was considered, for example in Refs. [15–19,23] and implies that the mass of microscopic black holes must be quantized and we must admit a minimum value [58] (for more general cases, see also Ref. [59]). Besides the conceptual relevance for the inclusion of gravity in a description of all forces of nature, there is also the

[5] The existence of this mass threshold may have phenomenological implications in models with extra spatial dimensions [48–52], where the fundamental (gravitational) length corresponds to energy scales potentially as low as a few TeV's.

potential phenomenological relevance of quantum mechanical effects during the formation of trapping horizons and black holes of astrophysical size.

In fact, one should not forget that the basic building blocks of matter remain the Standard Model particles, and that at such extreme energy regimes quantum effects should not be overlooked. All of the above conjectures would, therefore, be conspicuously substantiated if we could understand the extremely complex dynamics of colliding Standard Model particles, including the effect of the gravitational interaction, around the Planck scale [54–57,60]. To this purpose, the definition of the horizon wave function for simple spherical systems must be generalized to describe particle collisions and the inclusion of angular momentum in the initial and final configurations [61]. It appears to be hard to complete such steps without a more detailed model of "quantum black holes", to define the Hilbert space of the horizon wave function. One could, for example, incorporate the conjecture of Refs. [36–40] and describe the matter sourcing the black-hole geometry as a condensate at the phase transition.

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